

Quantum Weiss-Weinstein bounds for quantum metrology

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Sensing and imaging are among the most important applications of quantum information science. To investigate their fundamental limits and the possibility of quantum enhancements, researchers have for decades relied on the quantum Cramér-Rao lower error bounds pioneered by Helstrom. Recent work, however, has called into question the tightness of those bounds for highly nonclassical states in the non-asymptotic regime, and better methods are now needed to assess the attainable quantum limits in reality. Here we propose a new class of quantum bounds called quantum Weiss-Weinstein bounds, which include Cramér-Rao-type inequalities as special cases but can also be significantly tighter to the attainable error. We demonstrate the superiority of our bounds through the derivation of a Heisenberg limit and phase-estimation examples.

I. INTRODUCTION

Quantum noise is becoming a major limiting factor in sensing and imaging technology, with photon shot noise in particular imposing limits to modern gravitational-wave detectors [1] as well as optical microscopes [2]. Quantum metrology [3, 4], through the use of nonclassical states or innovative measurement schemes, promises to beat such conventional quantum limits and offer significant accuracy enhancements. This promise has led to renewed interest in the quantum estimation theory pioneered by Helstrom [5], and especially the quantum Cramér-Rao bounds (QCRBs) [5–7]. The bounds were originally developed to investigate thermal and laser sources, but they are now being applied to increasingly exotic quantum states for the purpose of quantum enhancements. The asymptotic attainability of Helstrom’s QCRB for one parameter [8, 9] and Holevo’s version for multiple parameters [10] suggests that the QCRBs can be tight; many proposals of quantum enhancements are based just on QCRBs [11–14]. Unfortunately, these works ignore the number of repeated trials needed to reach the asymptotic regime, and the requirement of many repetitions can negate the perceived advantage of their protocols, as realized by subsequent studies [15–23].

These mishaps suggest that Helstrom’s paradigm of quantum estimation theory can no longer fulfill the modern demands of quantum metrology and better approaches are needed to assess quantum sensors in highly nonclassical states. We can take inspiration from classical estimation theory, where it is common knowledge that Cramér-Rao-type inequalities can grossly underestimate the attainable estimation error [24]. Two new families of bounds have emerged there as the best candidates to supersede the Cramér-Rao family [24]: the Ziv-Zakai family [25] and the Weiss-Weinstein family [26, 27]. Although they are derived from distinct principles, both have been

found to remain remarkably tight to the attainable estimation errors in both non-asymptotic and asymptotic regimes, with diverse applications in engineering [24] as well as astronomy [28]. For the quantum problem, quantum Ziv-Zakai bounds (QZZBs) have recently been proposed and shown to be superior to QCRBs in many cases [15–17]. Although the QZZBs are trivial to prove and straightforward to evaluate, there is no general guarantee about their superiority over the QCRBs, so they have to be compared on a case-by-case basis. To overcome this problem, here we propose quantum versions of the Weiss-Weinstein bounds, which have the advantage of including Cramér-Rao-type inequalities as special cases. Through the derivation of a Heisenberg limit and examples of phase estimation, we further demonstrate that our new bounds can not only beat QCRBs but also QZZBs for tightness.

II. RESULTS

A. Quantum covariance inequality

Our quantum Weiss-Weinstein bounds (QWWBs) are based on a quantum generalization of the covariance inequality proposed by Weinstein and Weiss [24, 27]. It is a lower bound on the global estimation error matrix defined as

$$\Sigma := \int dx dy \epsilon(x, y) \epsilon(x, y)^\top p(x, y), \quad (1)$$

where $x \in \mathbb{R}^J$ is a column vector of J unknown parameters, y is the observation, $p(x, y)$ is their joint probability distribution, $\epsilon(x, y) := \hat{x}(y) - x$ is the error vector with respect to an estimator $\hat{x}(y)$, and \top denotes the matrix transpose. For the quantum problem [7],

$$p(x, y) = \text{Tr}[E_y \rho(x)], \quad (2)$$

where $\rho(x) = \rho_x p(x)$ is the hybrid density operator [29], ρ_x is the conditional density operator that models the quantum system as a function of x , $p(x)$ is the prior

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distribution, E_y is the positive operator-valued measure (POVM) that models the quantum measurement [5, 6], and Tr denotes the operator trace. Our quantum covariance inequality reads

$$\Sigma \geq CG^{-1}C^\top, \quad (3)$$

where G is a $K \times K$ real and strictly positive matrix defined as

$$G_{kk'} := \int dx \text{Re Tr} [L_k(x)^\dagger L_{k'}(x) \rho(x)] \quad (4)$$

in terms of a set of operators $\{L_k(x); k = 1, 2, \dots, K\}$ and C is a $J \times K$ real matrix defined as

$$C_{jk} := \int dx dy \epsilon_j(x, y) \text{Re Tr} [E_y L_k(x) \rho(x)]. \quad (5)$$

Equation (3) means that $\Sigma - CG^{-1}C^\top$ is positive-semidefinite. The proof of Eq. (3) is given in the Methods. To derive measurement-independent quantum bounds, we will choose a set of $L_k(x)$'s to make C independent of the POVM and the estimator.

B. Quantum Weiss-Weinstein bounds

Our QWWBs posit that each $L_k(x)$ satisfies

$$D_k(x) = \frac{1}{2} [L_k(x) \rho(x) + \rho(x) L_k(x)^\dagger], \quad (6)$$

$$D_k(x) := \frac{V_k(x + h_k) - V_k(x)}{|h_k|}, \quad (7)$$

$$V_k(x) := \mathcal{N}_k \rho(x)^{s_k} \circ \rho(x - h_k)^{1-s_k}, \quad (8)$$

where h_k is a real vector with length $|h_k|$ and the same dimension as that of x , $0 < s_k < 1$, $O_1 \circ O_2 := (O_1 O_2 + O_2 O_1)/2$ denotes the Jordan product, and \mathcal{N}_k is a normalization factor such that $\int dx \text{Tr} V_k(x) = 1$. This choice of $L_k(x)$ gives

$$C_{jk} = \frac{h_{kj}}{|h_k|}, \quad (9)$$

where h_{kj} is the j th component of h_k . To see this, notice that, with s_k being set in the range $(0, 1)$, $V_k(x)$ vanishes where $p(x)$ vanishes, leading to $\int dx V_k(x + h) = \int dx V_k(x)$ and $\int dx D_k(x) = 0$, as the domain of integration is \mathbb{R}^J and $p(x)$ must vanish at infinity. It then follows from Eqs. (5) and (6) and the completeness property of E_y that $C_{jk} = -\int dx x_j \text{Tr} [V_k(x + h) - V_k(x)]/|h_k|$. A change of variables gives $\int dx x_j V_k(x + h_k) = \int dx (x_j - h_{kj}) V_k(x)$, which leads to Eq. (9).

The QWWBs given by Eqs. (3)–(9) are applicable to any quantum measurement, any biased or unbiased estimator, and do not require ρ_x or $p(x)$ to be differentiable. They are a family of bounds that hold for any K , any h_k , and any $0 < s_k < 1$, such that tighter versions can be obtained by choosing these parameters

judiciously. The $|h_k| \rightarrow 0$ limit leads to the Bayesian QCRBs [7, 29] (see Appendix B), while finite h_k and $s_k \rightarrow 1$ lead to Bayesian multiparameter versions of the quantum bounds proposed by Tsuda and Matsumoto [30]. The classical Weiss-Weinstein bound is usually computed with $s_k = 1/2$ since it often maximizes the bound [24, 26, 27]; our examples later show that $s_k = 1/2$ can also lead to tight quantum bounds.

The $L_k(x)$ operators may not be uniquely determined by Eq. (6) for a given $\rho(x)$ and $D_k(x)$. We prove in Appendix C that the Hermitian $L_k(x)$'s give the tightest QWWB, though non-Hermitian choices may be easier to obtain in some cases. When $\rho(x)$ and $D_k(x)$ are of low rank, the following expression is useful to obtain the Hermitian $L_k(x)$'s:

$$L_k(x) = \sum_{\alpha, \beta | \lambda_\alpha + \lambda_\beta \neq 0} \frac{2 \langle \alpha | D_k(x) | \beta \rangle}{\lambda_\alpha + \lambda_\beta} |\alpha\rangle \langle \beta|, \quad (10)$$

where each $|\alpha\rangle$ is an eigenstate of $\rho(x)$ with eigenvalue λ_α . Taking $D_k(x)$ as the partial derivative with respect to x_k , Eq. (10) is a well-known expression for the symmetric logarithmic derivative operator [5, 31]. For non-Hermitian $L_k(x)$'s, the QWWB can be tightened by noting that $L_k(x) + i\alpha_k$, with α_k being an arbitrary real number, is also a solution of Eq. (6). Maximizing the positive matrix G over α_k leads to $\alpha_k = -\text{Im}\langle L_k(x) \rangle$, where $\langle \bullet \rangle := \int dx \text{Tr} [\bullet \rho(x)]$. We can therefore always replace G by $G - \Delta$ to tighten the QWWBs, where $\Delta_{kk'} = \text{Im}\langle L_k(x) \rangle \text{Im}\langle L_{k'}(x) \rangle$ is a positive-semidefinite matrix.

The QWWBs degenerates into the classical Weiss-Weinstein bounds [24, 26, 27] for a commuting family of ρ_x . In such a situation, we can identify a basis $\{|y\rangle\}$ in which all ρ_x are diagonal matrices, meaning that $\rho(x)$ can be equivalently expressed as a joint probability $p(x, y) := \langle y | \rho(x) | y \rangle$. Consequently, $L_k(x)$ is also diagonal with the basis $\{|y\rangle\}$, and can be expressed as a function

$$L_k(x, y) = \frac{\mathcal{N}_k}{|h_k|} \left\{ \left[\frac{p(x + h_k, y)}{p(x, y)} \right]^{s_k} - \left[\frac{p(x - h_k, y)}{p(x, y)} \right]^{1-s_k} \right\}, \quad (11)$$

where $\mathcal{N}_k = \mathbb{E}[p(x + h_k, y)^{s_k} / p(x, y)^{s_k}]^{-1}$, and $\mathbb{E}[\bullet]$ denotes the expectation value with respect to the joint probability $p(x, y)$. The Classical Weiss-Weinstein bound is still of the form Eq. (3) with C being given by Eq. (9), whereas G is expressed in a classical manner as $G_{kk'} = \mathbb{E}[L_k(x, y) L_{k'}(x, y)]$.

C. Single-parameter estimation

For single-parameter estimation, the error matrix reduces to the mean-square error $\Sigma = \int dx dy [\tilde{x}(y) - x]^2 p(x, y)$. The QWWBs become

$$\Sigma \geq \Sigma_W(s, h) := \frac{1}{\langle L(x)^\dagger L(x) \rangle - [\text{Im}\langle L(x) \rangle]^2}. \quad (12)$$

The following choice of $L(x)$ serves our purpose:

$$L(x) = \frac{\mathcal{N}(s, h)}{|h|} [\Lambda(s, h) - \Lambda(1 - s, -h)], \quad (13)$$

where $\Lambda(s, h) := \rho(x + h)^s \rho(x)^{-s}$ and $\mathcal{N}(s, h) = \langle \Lambda(s, h) \rangle^{-1}$. Here we use the convention that a power of a positive-semidefinite operator is taken only on its support [32]. $\rho(x)^{-1}$ is then the generalized inverse defined on the support and $\rho(x)^0$ is the projector onto the support. Consequently, $L(x)$ vanishes where $p(x)$ vanishes for $0 < s < 1$. Equation (12) becomes

$$\Sigma_W(s, h) = \frac{h^2 g(s, h)^2}{g(2s, h) + g(2 - 2s, -h) - 2\tilde{g}(s, 2h)}, \quad (14)$$

where $g(s, h) := \langle \Lambda(s, h) \rangle$ and $\tilde{g}(s, 2h) := \text{Re} \langle \Lambda(s, h)^\dagger \Lambda(1 - s, -h) \rangle$. When the conditional density operators ρ_x are of full rank, it can be shown that $\tilde{g}(s, h) = g(s, h)$. Equation (14) is then of the same form as the classical Weiss-Weinstein bound [26], but with a different function $g(s, h)$. Although the characteristics of $g(s, h)$ determines the QWWB in an intricate manner, some intuitive observations can be given as follows. The situation of particular interest is that $\Sigma_W(s, h)$ takes its maximum at a finite large value of h rather than at $h \rightarrow 0$, meaning that the QCRB underestimates the error. For the case of $\tilde{g}(s, h) = g(s, h)$, the denominator of Eq. (14) is bounded above by 2 due to $g(s, h) \in [0, 1]$. Then, considering the factor h^2 in the numerator, Eq. (14) may take its maximum at a finite large value of h , when $g(s, h)$ is not always far less than one as h becomes large. The estimation models with such a characteristic of $g(s, h)$ may be poorly assessed by only the QCRB, thereby are in need of the QWWB or the QZZB.

We now focus on phase estimation, a paradigmatic problem in quantum metrology. Assume $\rho_x = \exp(-ixH)\rho\exp(ixH)$, where ρ is the initial state and H is an Hermitian operator. In this case, $g(s, h)$ and $\tilde{g}(s, h)$ can be neatly separated as $g(s, h) = g_c(s, h)g_q(s, h)$ and $\tilde{g}(s, h) = g_c(s, h)\tilde{g}_q(s, h)$, where

$$g_c(s, h) = \int_{\{x: p(x) > 0\}} dx p(x + h)^s p(x)^{1-s} \quad (15)$$

is a classical component that depends only on the prior, and

$$g_q(s, h) = \text{Tr}(\rho_h^s \rho^{1-s}), \quad (16)$$

$$\tilde{g}_q(s, 2h) = \text{Re} \text{Tr}(\rho_h^s \rho_{-h}^{1-s} \rho^0) \quad (17)$$

are quantum components. If the initial state is pure, $\rho = |\psi\rangle\langle\psi|$, and since $\rho^s = \rho$ for a pure state, we obtain $g_q(s, h) = |z(h)|^2$ and $\tilde{g}_q(s, 2h) = \text{Re} z(h)^2 z(2h)^*$, where $z(h) := \langle\psi|\exp(-ihH)|\psi\rangle$. Interestingly, the quantity $g_q(s, h)$ also plays an important role in the quantum Chernoff bound for binary hypothesis testing [33–35], although no meaningful relationship between the Weiss-Weinstein bound and the Chernoff bound, apart from the coincidental mathematical similarity, has been discovered to our knowledge.

D. Heisenberg limit

The QWWBs can be used to derive a Heisenberg limit as follows. Let $|\psi\rangle = \sum_j c_j |j\rangle$ be a purification of the initial quantum state, where each $|j\rangle$ is an eigenvector of H with eigenvalue E_j . Then $\tilde{g}_q(s, 2h) = \sum_{jkl} |c_j c_k c_l|^2 \cos[h(E_j + E_k - 2E_l)]$. The cosine function can be bounded as $\cos\theta \geq 1 - \lambda|\theta|$, where $\lambda \approx 0.7246$ is the implicit solution of $\lambda = \sin\phi = (1 - \cos\phi)/\phi$ [15]. Thus, $\tilde{g}_q(s, 2h) \geq 1 - \lambda|h| \sum_{jkl} |c_j c_k c_l|^2 |E_j + E_k - 2E_l|$. Let E_0 be the minimum eigenvalue of H and $\Delta E_j := E_j - E_0$. By noting that $|E_j + E_k - 2E_l| = |\Delta E_j + \Delta E_k - 2\Delta E_l| \leq \Delta E_j + \Delta E_k + 2\Delta E_l$, it follows that $\tilde{g}_q(s, 2h) \geq 1 - 4\lambda|h|H_+$ with $H_+ := \text{Tr}(H\rho) - E_0$. Consequently, $\tilde{g}_q(s, 2h)$ is nonnegative when $|h| \leq 1/(4\lambda H_+)$; this implies that the QWWB is further bounded as

$$\Sigma \geq \Sigma'_W(h) := \kappa(h)h^2 |z(h)|^2 \text{ with} \quad \kappa(h) := \sup_{0 < s < 1} \frac{g_c(s, h)^2}{g_c(2s, h) + g_c(2 - 2s, -h)} \quad (18)$$

for $|h| \leq h_* := 1/(4\lambda H_+)$. The quantity $|z(h)|^2$ is the quantum fidelity between $|\psi\rangle$ and $\exp(-ihH)|\psi\rangle$, which is bounded as $|z(h)|^2 \geq 1 - |2h\lambda H_+|$ [15]. Taking $h = h_*$, one obtains

$$\Sigma \geq \Sigma_W(h_*)' \geq \frac{\kappa(h_*)}{32\lambda^2 H_+^2}. \quad (19)$$

We have not yet made any assumption about the prior, which is incorporated in $\kappa(h_*)$. Since $g_c(1/2, \pm h) \leq 1$, it follows that $\kappa(h) \geq g_c(1/2, h)^2/2$. The quantity $g_c(1/2, h)$, also known as the Bhattacharyya coefficient, measures the overlap between the prior probability distribution $p(x)$ and its displaced version $p(x + h)$. For a large enough H_+ (corresponding to a small enough h_*) such that $g_c(1/2, h_*) \approx 1$, Eq. (19) gives a Heisenberg limit as $1/(64\lambda^2 H_+^2)$, which is higher than the limit $1/(80\lambda^2 H_+^2)$ derived from a QZZB in Ref. [15]. Both this work and Ref. [15] use a linear lower bound on the fidelity; an even tighter Heisenberg limit can be obtained via the stronger fidelity bound in Ref. [20]. For a generator H with integer eigenvalues, a stronger Heisenberg limit was derived through some information-theoretic inequalities [21, 22].

E. Phase-estimation examples

We now demonstrate the tightness of QWWBs relative to other existing quantum bounds through two examples. The first example is the estimation of a random phase with Gaussian prior via a qubit. Assume that the initial qubit state is $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, the generator is $H = E|1\rangle\langle 1|$ with $E > 0$, and the standard deviation of the prior is σ . For this simple model, the minimum mean-square error (MMSE) can be analytically calculated [36, 37], and we can use it as a benchmark for the quantum

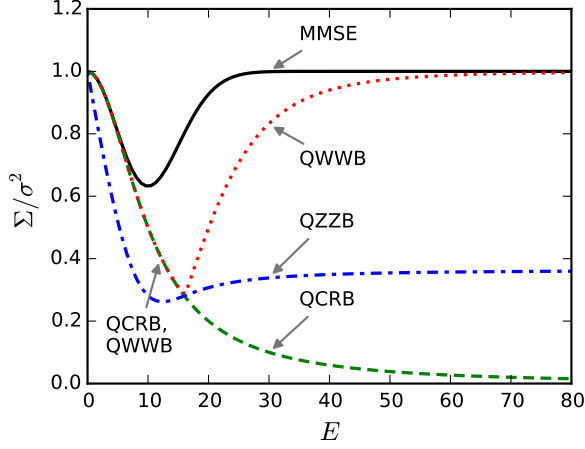


FIG. 1. Comparison of the MMSE (black solid), QWWB (red dotted), QZZB (blue dash-dotted), and the Bayesian QCRB (green dashed) for the estimation of a random phase via a qubit. The prior distribution is Gaussian with $\sigma = 0.1$ standard deviation. The QWWB is numerically optimized over $h \in [0, 10\sigma]$ while s is set to $1/2$. The MMSE and the error bounds are normalized with respect to the prior value σ^2 .

bounds. Setting $s = 1/2$, the QWWB is given by

$$\Sigma \geq \sup_h \frac{h^2 \exp[-h^2/(4\sigma^2)] \cos(hE/2)^2}{2 - 2 \exp[-h^2/(2\sigma^2) \cos(hE)]}, \quad (20)$$

see Appendix D for details. Since $\exp(-ixH)|\psi\rangle$ has a period of $2\pi/E$, x and $x + 2\pi/E$ are fundamentally indistinguishable from any quantum measurement. This ambiguity means that even the optimal measurement can produce an estimate in the wrong period, leading to substantial errors. The MMSE stays close to the prior value σ^2 as a result, as shown in Fig. 1. The QCRB, on the other hand, is incapable of accounting for the phase ambiguity because of its differential nature and severely underestimates the attainable error for large E . The QZZB is not much better, and the QWWB, being close to the QCRB where it is reasonably tight and also following the MMSE for larger E , is the clear winner in this benchmark example.

For the second example, we consider phase estimation using ν independent and identically distributed bosonic probes. For each probe, we assume $H = \sum_{j=0}^{\infty} j |j\rangle \langle j|$ and $|\psi\rangle = \sqrt{1-\epsilon}|0\rangle + \sqrt{\epsilon/M} \sum_{j=1}^M |j\rangle$, with $M \geq 1$ being an integer and $0 < \epsilon < 1$ [12]. In this case the MMSE is not known, and we have to rely on quantum bounds to investigate the fundamental limit. Figure 2 compares the three quantum bounds for $\epsilon = 0.1$ and $M = 10$. Though the asymptotic attainability of the QCRB [8, 9] means that it should be tight for large enough ν , the QCRB by itself is incapable of determining the ν needed for tightness. It is remarkable that the QWWB and the QZZB, though derived from different principles, follow similar behaviors here. Both are substantially higher than the

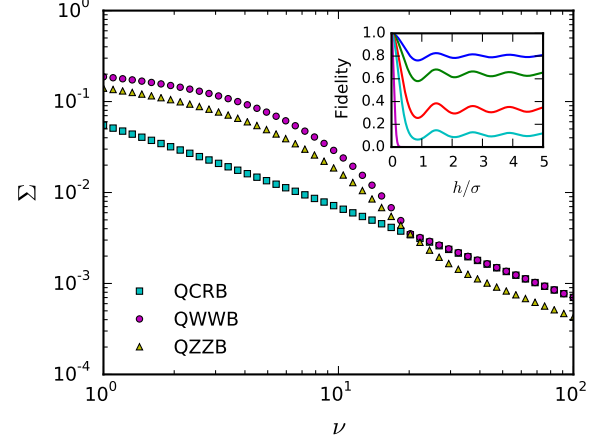


FIG. 2. Error bounds versus the number ν of identically distributed quantum bosonic probes. The prior distribution is Gaussian with $\sigma = 0.5$ standard deviation. The QWWB is numerically optimized over $h \in [0, 10\sigma]$ with $s = 1/2$, and the QZZB is computed according to Ref. [15]. The inset plots the fidelity $|\langle \psi | \exp(-ihH) | \psi \rangle|^2$ for $\nu = 1, 2, 5, 10, 100$ (from above to below).

QCRB for small ν and demonstrate a threshold behavior as ν is increased, revealing the regime where the prior information dominates and the QCRB is overly optimistic. Once again, the QWWB is higher than the other bounds for all values of ν .

F. Multiple test points

Similar to the classical Weiss-Weinstein family of error bounds [26, 27], the quantum bounds can be tightened by involving multiple test points. As an example, consider the QWWB for a single-parameter estimation with two test points, for which G is a 2×2 matrix whose inverse can be explicitly expressed as

$$G^{-1} = \frac{1}{\det G} \begin{pmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{pmatrix}. \quad (21)$$

Since $C = (1, 1)$, the lower bound from Eq. (3) becomes

$$CG^{-1}C^\top = \frac{G_{11} + G_{22} - G_{12} - G_{21}}{\det G} \quad (22)$$

$$\geq \max \left\{ \frac{1}{G_{11}}, \frac{1}{G_{22}} \right\}, \quad (23)$$

where the inequality is due to the fact that G is symmetric and positive, meaning that the two-test-point lower bound is tighter than that given by either of the two test points. Following the same strategy that derives the combined Bayesian bound in classical parameter estimation [24], we can set the first test point h_1 to an infinitesimal value and the second test point h_2 to a finite value, leading to a combined quantum error bound tighter than the QCRB.

III. DISCUSSION

Our QWWBs set a higher standard in quantum metrology. Not only do they include QCRBs as special cases and inherit their asymptotic tightness at least for one parameter [8, 9], they can also beat the recently invented QZZBs [15] and serve as the more natural successors of the Cramér-Rao family in the post-Helstrom era of quantum metrology. Our results demonstrate that differential geometry of quantum states alone [31, 38] cannot guarantee their usefulness; more general distance measures, such as the quantum Chernoff distance used in our QWWBs and the trace distance in the QZZBs, should be consulted to establish tighter quantum limits to parameter estimation, especially for nonclassical states or nontrivial parameter dependence. Future proposals of quantum metrological schemes should no longer rely only on QCRBs to support their cases without also investigating their tightness. We envision our QWWBs to be the new standard against which these proposals should be assessed.

ACKNOWLEDGMENTS

We acknowledge helpful discussions with Ranjith Nair, Shan Zheng Ang, and Shilin Ng. This work is supported by the Singapore National Research Foundation under NRF Grant No. NRF-NRFF2011-07 and Singapore Ministry of Education Academic Research Fund Tier 1 Project R-263-000-C06-112.

CONTRIBUTIONS

X.-M. L. invented the quantum Weiss-Weinstein bounds presented here and performed all the proofs and calculations. M. T. conceived the problem. Both authors discussed extensively during the course of this work and contributed to the writing of the manuscript.

Appendix A: Proof of the quantum covariance inequality

Here we prove Eq. (3). Let u and v be arbitrary real column vectors of dimension J and K respectively. It follows from the definitions that

$$u^\top \Sigma u = \int dx dy \epsilon_u(x, y)^2 \text{Tr}(A^\dagger A), \quad (\text{A1})$$

$$v^\top G v = \int dx dy \text{Tr}(B^\dagger B), \quad (\text{A2})$$

where $\epsilon_u(x, y) := \sum_j u_j \epsilon_j(x, y)$, $A := \sqrt{E_y} \sqrt{\rho(x)}$, and $B := \sqrt{E_y} [\sum_k v_k L_k(x)] \sqrt{\rho(x)}$. In Eq. (A2), we have used $\int dy E_y = I$ with I being the identity operator. As

a result of the Cauchy-Schwarz inequality,

$$\sqrt{u^\top \Sigma u v^\top G v} \geq \int dx dy |\epsilon_u(x, y)| \sqrt{\text{Tr}(A^\dagger A) \text{Tr}(B^\dagger B)}. \quad (\text{A3})$$

From the inequality $\text{Tr}(A^\dagger A) \text{Tr}(B^\dagger B) \geq |\text{Tr}(A^\dagger B)|^2$ followed by $|\text{Tr}(A^\dagger B)| \geq |\text{Re Tr}(A^\dagger B)|$, we get

$$\begin{aligned} (u^\top \Sigma u) (v^\top G v) &\geq \left[\int dx dy |\epsilon_u(x, y) \text{Re Tr}(A^\dagger B)| \right]^2 \\ &\geq \left| \int dx dy \epsilon_u(x, y) \text{Re Tr}(A^\dagger B) \right|^2 = (u^\top C v)^2. \end{aligned} \quad (\text{A4})$$

Taking $v = G^{-1} C^\top u$ implies $(u^\top \Sigma u) (u^\top C G^{-1} C^\top u) \geq (u^\top C G^{-1} C^\top u)^2$. Since G is strictly positive, $u^\top C G^{-1} C^\top u$ is positive, leading to $u^\top \Sigma u \geq u^\top C G^{-1} C^\top u$. As this inequality holds for any real vector u , Eq. (3) results.

Appendix B: Relation between QWWB and QCRB

We here show that the QWWBs include the QCRB as a special case. Let h_k be along the direction of x_k in the parameter vector space. Suppose that $\rho(x)$ is differentiable. When $|h_k| \rightarrow 0$, one has

$$\rho(x + h_k)^{s_k} \simeq \rho(x)^{s_k} + |h_k| \partial \rho(x)^{s_k} / \partial x_k. \quad (\text{B1})$$

It can be shown from Eqs. (7), (8) and (B1) that $D_k(x) \simeq \mathcal{N}_k \partial \rho(x) / \partial x_k$, where the normalizing factor \mathcal{N}_k^{-1} can be given by

$$\mathcal{N}_k^{-1} \simeq 1 - |h_k| \int dx \text{Tr} \left[\rho(x)^{s_k} \frac{\partial \rho(x)^{1-s_k}}{\partial x_k} \right]. \quad (\text{B2})$$

Let $\rho(x) = \sum_\alpha \lambda_\alpha |\phi_\alpha\rangle \langle \phi_\alpha|$ be the eigenvalue decomposition. Since $\rho(x)^{s_k} = \sum_\alpha \lambda_\alpha^{s_k} |\phi_\alpha\rangle \langle \phi_\alpha|$, it follows that

$$\begin{aligned} &\int dx \text{Tr} \left[\rho(x)^{s_k} \frac{\partial \rho(x)^{1-s_k}}{\partial x_k} \right] \\ &= \sum_\alpha \int dx \left[\lambda_\alpha^{s_k} \frac{\partial \lambda_\alpha^{1-s_k}}{\partial x_k} + \lambda_\alpha \left(\langle \phi_\alpha | \frac{\partial \phi_\alpha}{\partial x_k} \rangle + \langle \frac{\partial \phi_\alpha}{\partial x_k} | \phi_\alpha \rangle \right) \right] \\ &= (1 - s_k) \sum_\alpha \int dx \frac{\partial \lambda_\alpha}{\partial x_k} + \sum_\alpha \int dx \lambda_\alpha \frac{\partial}{\partial x_k} \langle \phi_\alpha | \phi_\alpha \rangle \\ &= 0, \end{aligned} \quad (\text{B3})$$

where we have used $\lambda_\alpha|_{x_k=\pm\infty} = 0$ in the last equality. Thus, $D_k(x) \simeq \partial \rho(x) / \partial x_k$ when $|h_k| \rightarrow 0$. Consequently, the operator $L_k(x)$ becomes the symmetric logarithmic derivative operator (not necessarily to be Hermitian, see Ref. [29]) for $\rho(x)$ with respect to x_k , and the resulting QWWB becomes a corresponding QCRB.

Appendix C: Hermitian $L_k(x)$ tightening the QWWB

Here, we prove that the Hermitian $L_k(x)$ gives the tightest lower bound on the estimation-error covariance matrix among all choices of $L_k(x)$ satisfying Eq. (6) for given $\rho(x)$ and $D_k(x)$. This can be seen from the following Proposition.

Proposition. Suppose that L is an operator satisfying

$$\frac{1}{2}(L\rho + \rho L^\dagger) = D, \quad (\text{C1})$$

where ρ is a given positive semidefinite operator and D is a given Hermitian operator. Then,

$$\min \text{Tr}(L^\dagger L \rho) = \text{Tr}(\tilde{L}^\dagger \tilde{L} \rho), \quad (\text{C2})$$

where the minimum is taken over all solutions of Eq. (C1) for L , and \tilde{L} denotes a Hermitian solution.

Proof. Let $M = (L + L^\dagger)/2$ and $N = (L - L^\dagger)/(2i)$, which are both Hermitian operators. Let $\rho = \sum_j \lambda_j |j\rangle\langle j|$ be the eigenvalue decomposition. It follows from Eq. (C1) and $L = M + iN$ that

$$D_{jk} = \frac{1}{2}(\lambda_k + \lambda_j)M_{jk} + \frac{i}{2}(\lambda_k - \lambda_j)N_{jk}, \quad (\text{C3})$$

where the elements of the matrices are represented in the basis $\{|j\rangle\}$. This equality implies that we can always freely choose N and determine M accordingly in terms of D and N . When $\lambda_j + \lambda_k \neq 0$, we have

$$M_{jk} = \frac{2D_{jk} + i(\lambda_j - \lambda_k)N_{jk}}{\lambda_j + \lambda_k}, \quad (\text{C4})$$

which implies that $L_{jk} = (2D_{jk} + 2i\lambda_j N_{jk})/(\lambda_j + \lambda_k)$. It then follows that

$$\text{Tr}(L^\dagger L \rho) = \sum_{j,k|\lambda_k>0} \lambda_k |L_{jk}|^2 \quad (\text{C5})$$

$$= \sum_{j,k|\lambda_k>0} \frac{4\lambda_k}{(\lambda_j + \lambda_k)^2} |D_{jk} + i\lambda_j N_{jk}|^2 \quad (\text{C6})$$

$$= \sum_{j,k|\lambda_k>0} A_{jk} + \sum_{j,k|\lambda_j>0, \lambda_k>0} B_{jk}, \quad (\text{C7})$$

where

$$A_{jk} := \frac{4\lambda_k}{(\lambda_j + \lambda_k)^2} (|D_{jk}|^2 + \lambda_j^2 |N_{jk}|^2), \quad (\text{C8})$$

$$B_{jk} := \frac{4i\lambda_j \lambda_k}{(\lambda_j + \lambda_k)^2} (D_{jk}^* N_{jk} - D_{jk} N_{jk}^*). \quad (\text{C9})$$

Since both D and N are Hermitian, we have $D_{jk}^* = D_{kj}$ and $N_{jk}^* = N_{kj}$, which implies that the matrix B is anti-symmetric as $B_{jk} = -B_{kj}$. Therefore,

$$\text{Tr}(L^\dagger L \rho) = \sum_{j,k|\lambda_k>0} A_{jk} \geq \sum_{j,k|\lambda_j>0} \frac{4\lambda_k |D_{jk}|^2}{(\lambda_j + \lambda_k)^2}. \quad (\text{C10})$$

The equality in the above inequality holds when all N_{jk} vanishes, meaning that L is Hermitian. \square

Now, let us consider the case where each $L_k(x)$ may be non-uniquely determined by $D_k(x)$ and $\rho(x)$ through Eq. (6). Denote the Hermitian solution for $L_k(x)$ by $\tilde{L}_k(x)$ and define the matrix \tilde{G} by $\tilde{G}_{kk'} = \text{Re Tr}[\tilde{L}_k(x)^\dagger \tilde{L}_{k'}(x) \rho(x)]$. Let u be an arbitrary real vector. In terms of the above Proposition with $D = \sum_k u_k D_k(x)$ and $L = \sum_k u_k L_k(x)$, it can be shown that $u^\top G u \geq u^\top \tilde{G} u$, thus $G \geq \tilde{G}$. Suppose that G is strictly positive, then $G \geq \tilde{G}$ implies $\tilde{G}^{-1} \geq G^{-1}$. Thus, the Hermitian $L_k(x)$ give the tightest lower bound in the Weiss-Weinstein family.

Appendix D: Phase-estimation example

Here, we calculate the MMSE, the QWWB, the QCRB, and the QZZB for the first example in Sec. II E. For a unitary sensing $U_x = \exp(-ixH)$ and a Gaussian prior distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad (\text{D1})$$

the MMSE of estimating x is given by [36]

$$\Sigma_{\min} = \sigma^2 - \sigma^4 \mathcal{F}(\bar{\rho}, H), \quad (\text{D2})$$

where $\mathcal{F}(\bar{\rho}, H)$ is the quantum Fisher information about a parameter θ in the parametric quantum state $U_\theta \bar{\rho} U_\theta^\dagger$, where $\bar{\rho} := \int_{-\infty}^{\infty} dx p(x) U_x \rho U_x^\dagger$ with ρ being the initial state. With the eigenvalue decomposition $\bar{\rho} = \sum_j \lambda_j |j\rangle\langle j|$, one has

$$\mathcal{F}(\bar{\rho}, H) = \sum_{j,k|\lambda_j+\lambda_k>0} \frac{2(\lambda_j - \lambda_k)^2 |H_{jk}|^2}{\lambda_j + \lambda_k}. \quad (\text{D3})$$

In our example, the initial state is $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and the generator of the unitary sensing is $H = E|1\rangle\langle 1|$, where E is a positive number. Then, the average state is given by

$$\bar{\rho} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) + \frac{\gamma}{2}(|0\rangle\langle 1| + |1\rangle\langle 0|) \quad (\text{D4})$$

with $\gamma := \exp(-E^2\sigma^2/2)$. The eigenvalues and eigenvectors of $\bar{\rho}$ are $(1 \pm \gamma)/2$ and $(|0\rangle \pm |1\rangle)/\sqrt{2}$ respectively. It then follows from Eq. (D3) that $\mathcal{F}(\bar{\rho}, H) = \gamma^2 E^2$, which implies

$$\Sigma_{\min} = \sigma^2 - \sigma^4 E^2 \exp(-E^2\sigma^2). \quad (\text{D5})$$

To obtain the QWWB, one only needs $g_c(s, h) = \exp[-h^2 s(1-s)/(2\sigma^2)]$ and $z(h) = (1 + e^{-iEh})/2$, with which the QWWB is given by

$$\Sigma_W(s, h) = \frac{h^2 g_c(s, h)^2 |z(h)|^4}{[g_c(2s, h) + g_c(2 - 2s, -h)] |z(h)|^2 - 2g_c(s, 2h) \operatorname{Re} z(h)^2 z(2h)^*}. \quad (\text{D6})$$

Taking $s = 1/2$ for simplicity, we obtain the QWWB optimized over h as follows:

$$\Sigma_W = \sup_h \frac{h^2 \exp(-\frac{h^2}{4\sigma^2}) \cos(\frac{hE}{2})^2}{2 - 2 \exp(-\frac{h^2}{2\sigma^2}) \cos(hE)}. \quad (\text{D7})$$

After some algebras, the QCRB is given by

$$\Sigma_C = \frac{1}{1/\sigma^2 + E^2}, \quad (\text{D8})$$

and the QZZB is given by

$$\Sigma_Z = \frac{1}{2} \int_0^{+\infty} dh h \operatorname{erfc}\left(\frac{h}{2\sqrt{2}\sigma}\right) [1 - \sqrt{1 - |z(h)|^2}], \quad (\text{D9})$$

where $\operatorname{erfc}(x) = (2/\sqrt{\pi}) \int_x^{+\infty} dt e^{-t^2}$ is the complementary error function.

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